7- Root Locus

The transient response of a system plays an important role in the design of Control system. The nature of the transient response is determined by the location of poles of the closed loop system. Usually the loop gain of the system is adjustable and the value of this gain determines the location of poles of the closed loop ~~gai~~ system. It will be very informative if we can determine how these poles change their location as the gain is increased. The locus of these roots ~~are~~ as one parameter of the system, usually the gain, is varied over a wide range, is known as the root locus plot of the system

7-1 Basic Idea:-

Consider the characteristic equation of a second order system, given by

$$s^2 + as + K = 0$$

a: is a constant.

K: is the variable.

we would like to obtain the locus of the roots, as K is changed from 0 to ∞.

The roots of the characteristic equation are given by,

$$S_{1,2} = -\frac{a}{2} \mp \sqrt{\frac{a^2-4K}{2}}$$

- if $K=0 \Rightarrow S_{1,2} = -\frac{a}{2} \mp \frac{a}{2}$

$$\left.\begin{aligned} S_1 &= -\frac{a}{2}+\frac{a}{2} \Rightarrow S_1 = 0 \\ S_2 &= -\frac{a}{2}-\frac{a}{2} \Rightarrow S_2 = -a \end{aligned}\right\} \text{real part poles}$$

- if $0 < K < \frac{a^2}{4}$, let $K = \frac{a^2}{8}$

$$S_{1,2} = -\frac{a}{2} \mp \frac{\sqrt{a^2-4\frac{a^2}{8}}}{2}$$

$$= -\frac{a}{2} \mp \frac{\sqrt{\frac{2a^2-a^2}{2}}}{2}$$

$$= -\frac{a}{2} \mp \frac{a}{2\sqrt{2}}$$

$$\left.\begin{aligned} S_1 &= \frac{-1.4a+a}{2.8} \Rightarrow S_1 = \frac{-0.4a}{2.8} = -0.14a \\ S_2 &= \frac{-1.4a-a}{2.8} \Rightarrow S_2 = \frac{-2.4a}{2.8} = -0.857 \end{aligned}\right\} \text{real part poles}$$

- $a^2 - 4K > 0 \Rightarrow K < \frac{a^2}{4}$

if $K = \frac{a^2}{4} \Rightarrow s_{1,2} = -\frac{a}{2}$ just on pole

- if $K > \frac{a^2}{4}$, let $K = \frac{a^2}{2}$

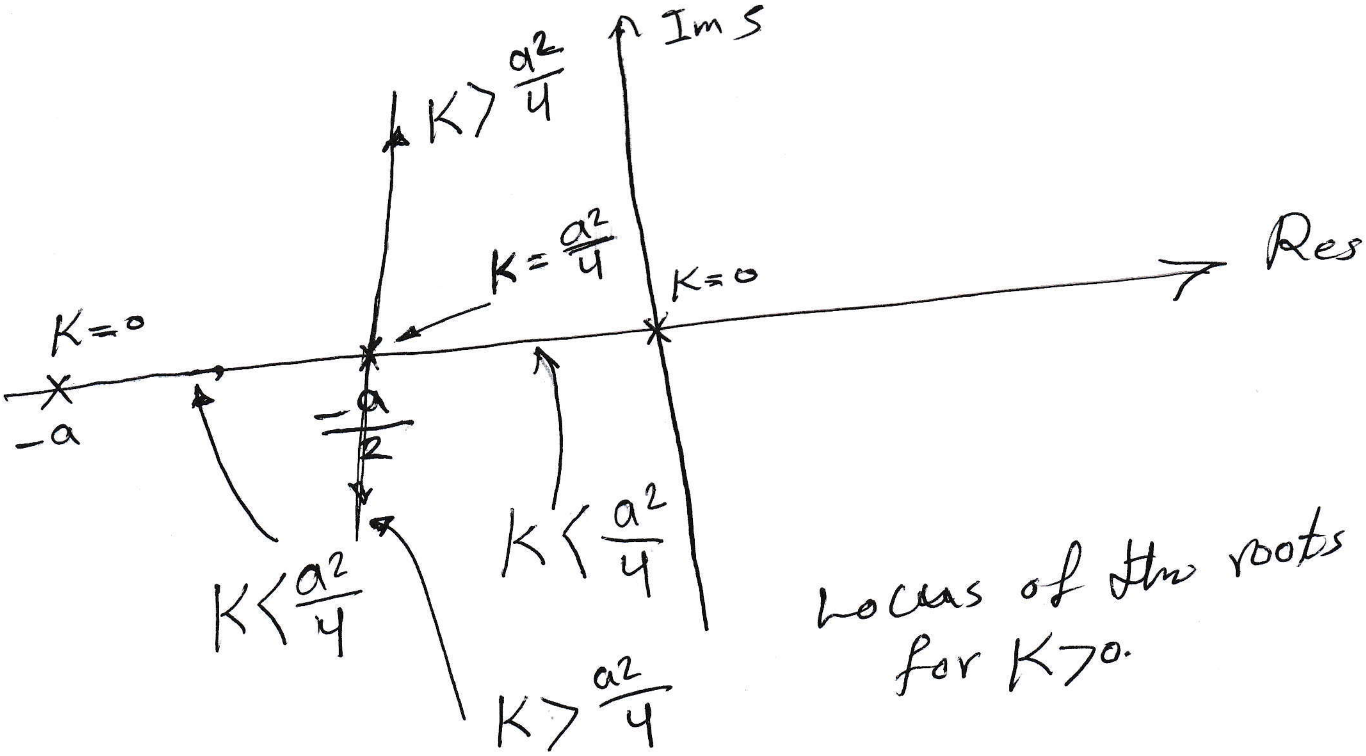
$$S_{1,2} = -\frac{a}{2} \mp \frac{\sqrt{a^2 - 4 * \frac{a^2}{2}}}{2}$$

$$S_{1,2} = -\frac{a}{2} \mp \frac{\sqrt{-a^2}}{2}$$

$$S_{1,2} = -\frac{a}{2} \mp j\frac{a}{2}$$

$$s_1 = -\frac{a}{2} + j\frac{a}{2}, \quad S_2 = -\frac{a}{2} - j\frac{a}{2}$$

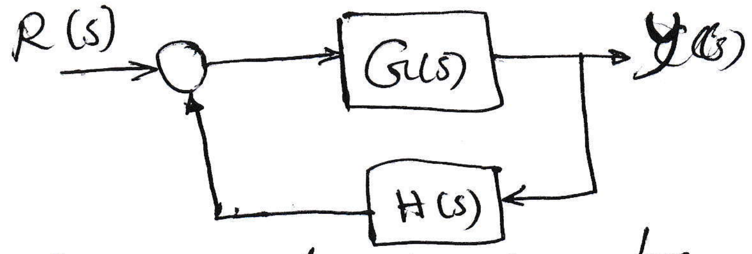
There are two imajinary poles on s-plane.



Locus of the roots for $K > 0$.

7-2 Root Locus technique :-

Consider the closed Loop system show in figure below



The closed loop transfer function is given by

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

the characteristic equation is

$$D(s) = 1 + G(s)H(s) = 0$$

$$G(s)H(s) = -1$$

- Since s is a complex variable, G(s) H(s) is a complex quantity and it will be

$$|G(s)H(s)| = 1 \qquad \text{(magnitude)}$$

and $\quad \angle G(s)H(s) = \pm 180(2K+1), \quad K = 0, 1, 2, \ldots$ (angle).

The loop transfer function G(s) H(s) with a gain parameter K, as

$$G(s)H(s) = \frac{K(s+Z_1)(s+Z_2)\cdots(s+Z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)}$$

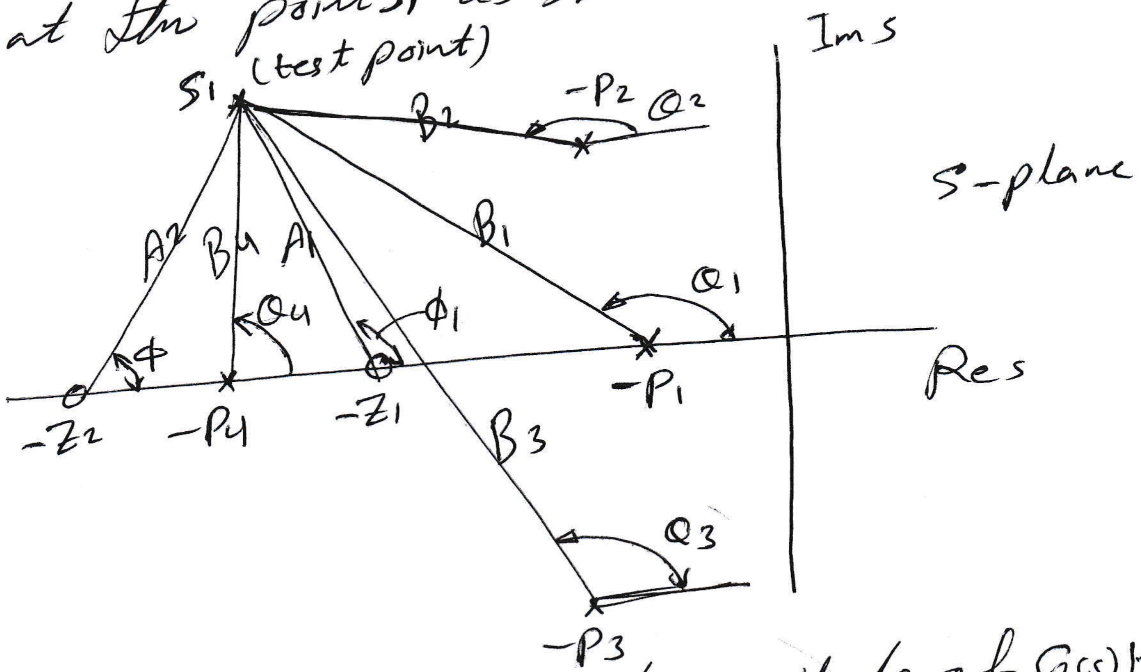
the characteristic equation becomes,

$$1+G(s)H(s) = 1+ \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0$$

$s=-z_i$ for $i=1,2,\ldots m$ are the open loop zeros

$s=-p_i$ for $i=1,2,\ldots n$ are the open loop poles.

To check the system, a point $s=s_1$ satisfies angle criterion or not, measure the angles made by open loop poles and zeros at the point s_1 as shown below.



We can easily obtain the angle and magnitude of $G(s)H(s)$ for $s=s_1$ as

$$\angle G(s_1)H(s_1) = \phi_1+\phi_2-\theta_1-\theta_2-\theta_3-\theta_4$$

$$|G(s)H(s)| = \frac{A_1A_2}{B_1B_2B_3B_4}$$

Properties of root locus :-

1- The root locus is symmetrical about the real axis.

2- There are n root locus branches each starting from an open loop pole for K=0. m of these branches terminate on m open loop zeros. the remaining n−m branches go to zero at infinity.

3- The n−m branches going to zeros at infinity, do so along asymptotes making angles

$$\phi = \frac{(2K \pm 1)180}{n-m} ; \quad K = 0, 1, 2, \ldots (n-m-1)$$

$n-m=1$ $\quad \phi = 180^\circ$

$n-m=2$ $\quad \phi = 90, -90$

$n-m=3$ $\quad \phi = 60, 180, -60$

$n-m=4$ $\quad \phi = 45, 135, -135, -45$ and so on.

4- The asymptotes meet the real axis at

$$\sigma = \frac{\sum \text{real parts of poles} - \sum \text{real parts of zeros}}{n-m}$$

5- Segments of real axis are parts of root locus if the total number of real poles and zeros together to their right is odd.

6- Breakaway or Breaking points

These are points in s-plane where multiple closed loop poles occur. These are the roots of the equation,

$$\frac{dK}{ds} = 0$$

only those roots which satisfy the angle criterion also, are the breakaway or break points. If "r" root locus branches break away at a point on real axis, the breakaway directions are given by $\pm \frac{180^\circ}{r}$.

7- The angle of departure of the root locus at a complex pole is given by,

$$\phi_p = \pm (2K+1)180 + \phi$$

Where ϕ is the net angle contributed by all other open loop zeros and poles at this pole.

The angle of arrival at a complex zero is given by

$$\phi_z = \pm (2K+1)180 - \phi$$

ϕ: is the angle contributed by all other open loop poles and zeros at this zero.

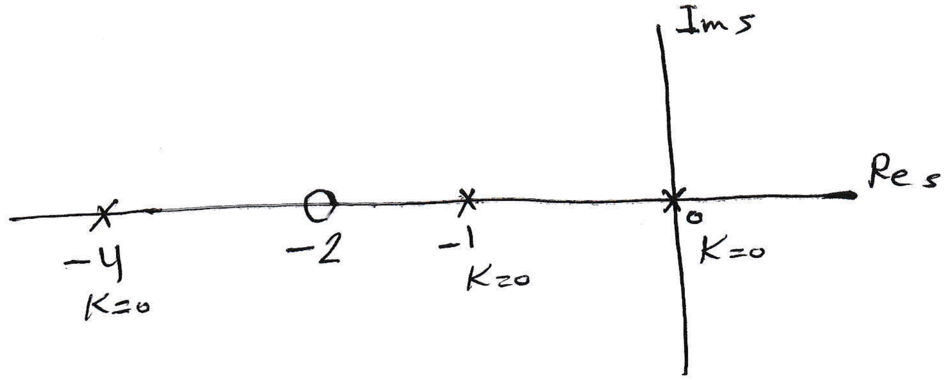
8- the cross over point of the root locus on the imaginary axis is obtained by using Routh Hurwitz Criterion.

EX

Sketch the root locus of a unity feedback system with

$$G(s) = \frac{K(s+2)}{s(s+1)(s+4)}$$

Step 1



Step 2

there are 3 root locus branches because we have 3 poles, starting from $s=0$, $s=-1$, $s=-4$. One of the root locus branches approaches this zero for $K=\infty$.

Step 3 Angles made by asymptotes.

$$\phi = \frac{2K+1}{n-m} 180 \quad , K=0,1,2,\cdots (n-m-1)$$

$n=3$, $m=1$, there are $n-m=2$ asymptotes along which the root locus branches go to infinity

$$\phi_1 = \frac{180}{2} = 90^\circ \quad , K=0$$

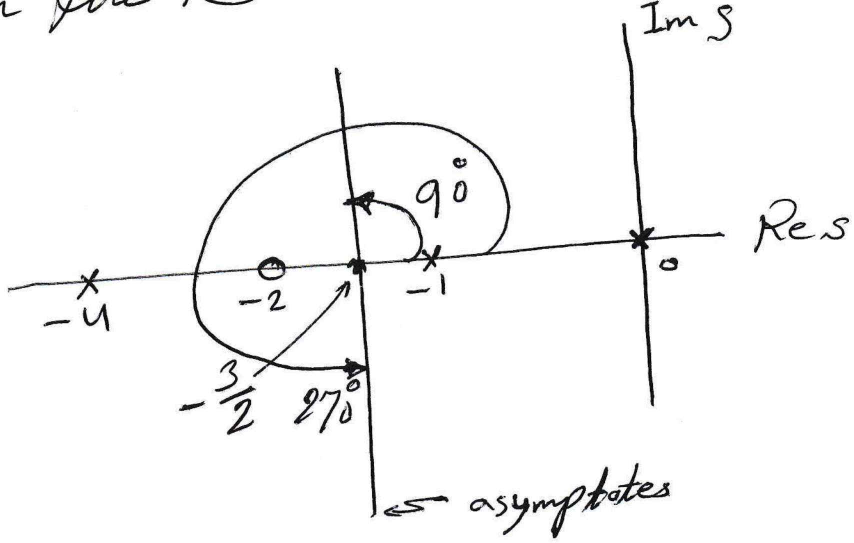
$$\phi_2 = \frac{3 \times 180}{2} = 270 \quad , K=1$$

Step 4 Centroid

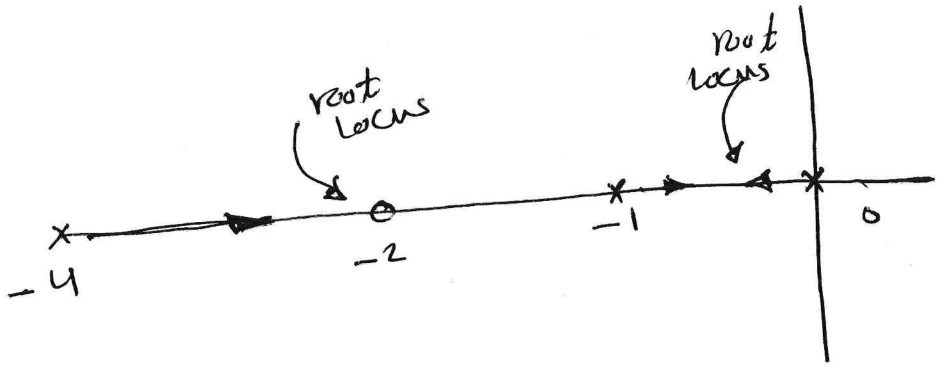
$$\sigma = \frac{\sum \text{no of Poles} - \sum \text{no of Zeros}}{n-m}$$

$$= \frac{0-1-4-(-2)}{2} = -\frac{3}{2}$$

Draw the asymptotes making angles 90° and 270° at $\sigma = -\frac{3}{2}$ on the real axis as shown below



Step 5 Segment



Step 6

There is one breakaway point between $s=0$ and $s=-1$.

$$G(s)H(s) = \frac{K(s+2)}{s(s+1)(s+4)}$$

or by characteristic equation

$$1 + \frac{K(s+2)}{s(s+1)(s+4)} = 0$$

$$\frac{K(s+2)}{s(s+1)(s+4)} = -1$$

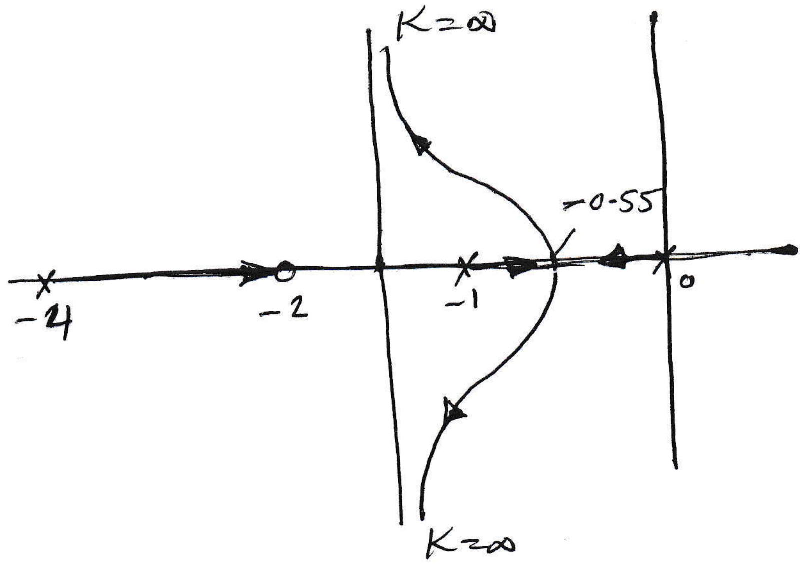
$$K = -\frac{s(s+1)(s+4)}{(s+2)}$$

$$\frac{dK}{ds} = -\frac{(s+2)\left[3s^2+10s+4\right] - s(s+1)(s+4)}{(s+2)^2} = 0$$

$$2s^3 + 11s^2 + 20s + 8 = 0$$

By trial and error we can find the breakaway point between $s=0$ and $s=-1$

$s = -0.55$



EX Sketch the root locus of the system with loop transfer function

$$G(s)H(s) = \frac{K}{s(s+2)(s^2+s+1)}$$

step 1 Plot the poles and zeros of $G(s)H(s)$

Zero: No

poles: $0, -2, -0.5 \pm j\frac{\sqrt{3}}{2}$

step 2

there are 4 root locus branches.

step 3 Angle of asymptotes

$$\phi = \frac{(2K+1)180}{n-m} \qquad K=0,1,2,3$$

$$\phi = \frac{180}{4} = 45^\circ$$

$$\phi_2 = \frac{3*180^\circ}{4} = 135^\circ$$

$$\phi_3 = \frac{5*180^\circ}{4} = 225^\circ$$

$$\phi_4 = \frac{7*180^\circ}{4} = 315^\circ$$

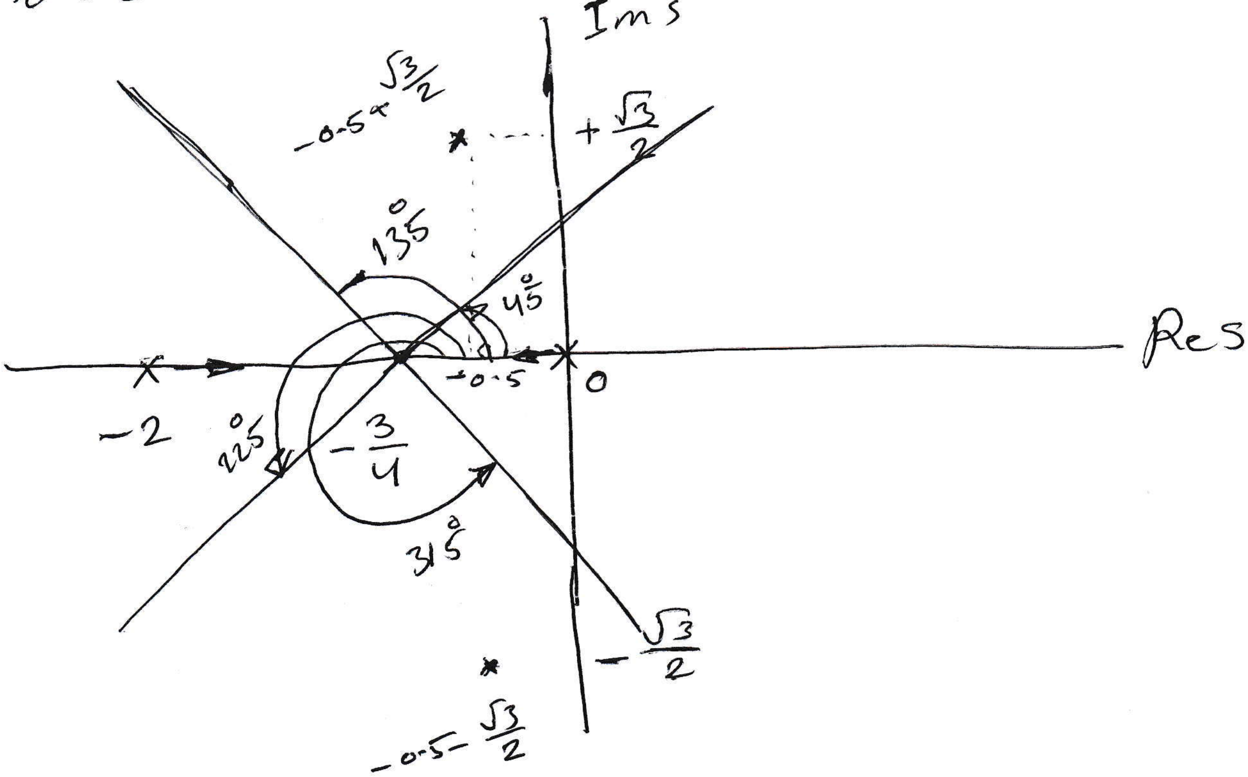
Step 4 Centroid

$$\sigma = \frac{\Sigma \text{real parts of poles} - \Sigma \text{real parts of zeros}}{n-m}$$

$$= \frac{0-2-0.5-0.5}{4} = -\frac{3}{4}$$

Step 5 Root locus on real axis.

Root locus lies between 0 and -2.



Step 6 Breakaway points

the characteristic equation is

$$1 + GH = 0$$

$$1 + \frac{K}{s(s+2)(s^2+s+1)} = 0$$

$$K = -s(s+2)(s^2+s+1)$$

$$= -s^4 - 3s^3 - 3s^2 - 2s$$

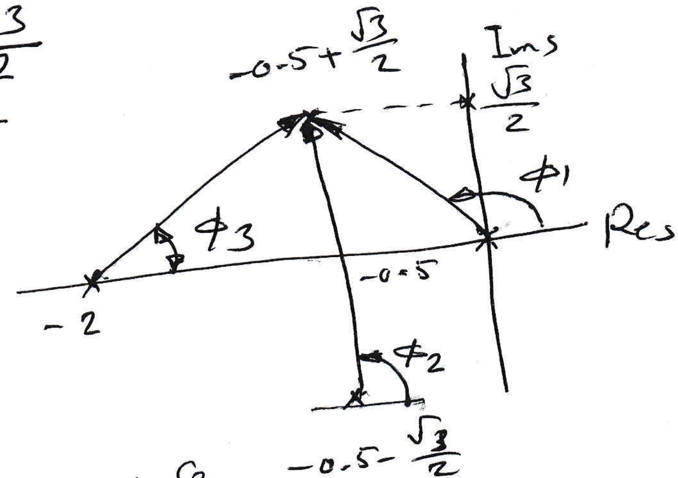
$$\frac{dK}{ds} = -4s^3 - 9s^2 - 6s - 2 = 0$$

$$4s^3 + 9s^2 + 6s + 2 = 0$$

solving this for a root in the range 0 to -2 by trial and error. $\therefore$ $s = -1.455$

Step 7 Angle of departure from complex poles

$$s = -0.5 \pm j\frac{\sqrt{3}}{2}$$



Draw vectors from all other poles and zero to complex pole $s = -0.5 \pm j\frac{\sqrt{3}}{2}$

$$\phi = -\phi_1 - \phi_2 - \phi_3$$

$$= -\left(180 - \tan^{-1}\frac{\frac{\sqrt{3}}{2}}{0.5}\right) - 90 - \tan^{-1}\frac{\frac{\sqrt{3}}{2}}{1.5}$$

$$= -120 - 90 - 30 = -240$$

Angle of departure

$$\phi_p = \pm(2K+1)180 + \phi \qquad K=0,1,2$$

$$= 180 - 240$$

$$= -60$$

Step 8

Crossing of $j\omega$ -axis

The characteristic equatio is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+2)(s^2+s+1)} = 0$$

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

By Routh Criterion

s^4	1	3	K
s^3	3	2	
s^2	$\frac{7}{3}$	K	
s^1	$\frac{\frac{14}{3} - 3K}{7/3}$		
s^0	K		

$$\frac{14}{3} = 3K \Rightarrow K = \frac{14}{9}$$

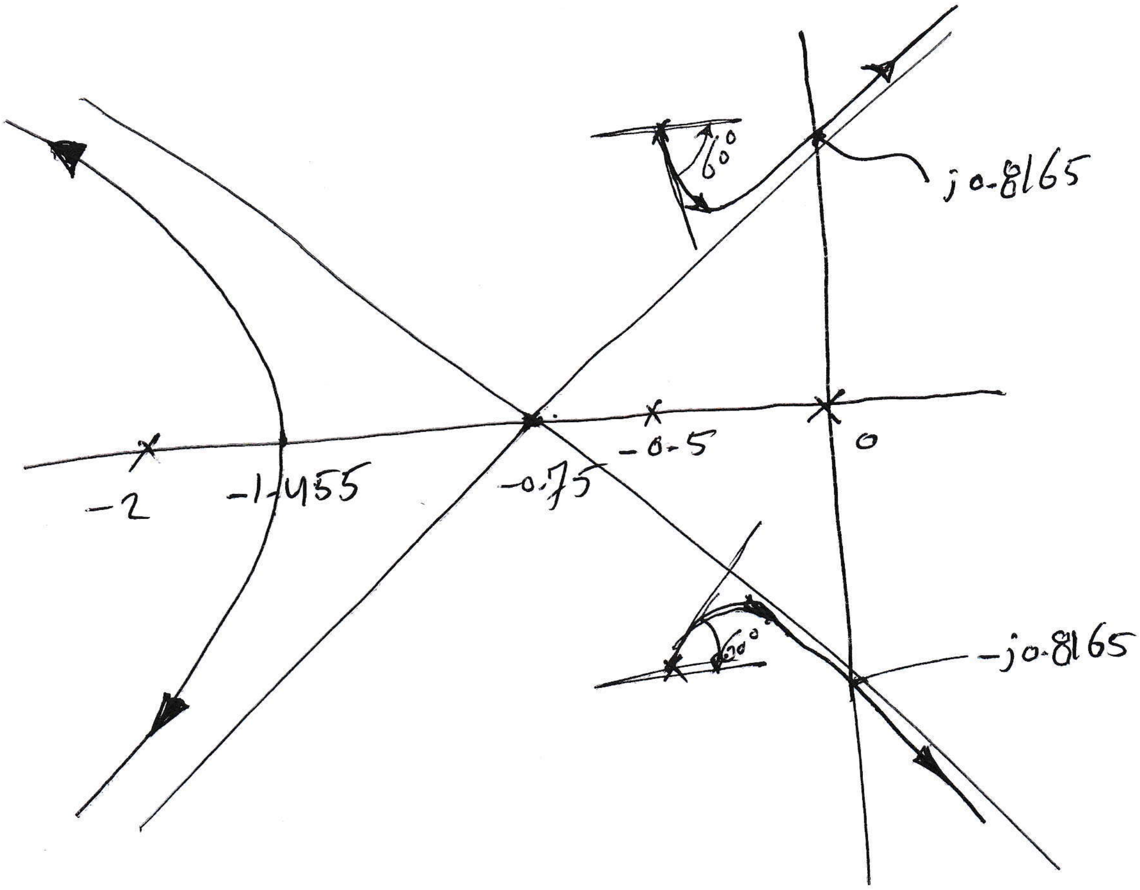
Auxiliary equation for this value of K is,

$$\frac{7}{3}s^2 + \frac{14}{9} = 0$$

$$s^2 = -\frac{2}{3}$$

$$s = \pm j\sqrt{\frac{2}{3}} = \pm\, 0.8165$$

The root locus cross $j\omega$-axis at $s = \pm j\,0.8165$ for $K = \frac{14}{9}$



EX Sketch the root locus of the following unity feedback system with

$$G(s) = \frac{K}{s(s+2)(s^2+2s+4)}$$

a) Find the value of K at breakaway points.

b) Find the value of K and the closed loop poles at which the damping factor is 0.6.

step 1 Plot the poles and zeros

Zeros : No

poles : $0, -2, -1 \pm j\sqrt{3}$

step 2 there are 4 root locus branches starting from the open loop plots.

step 3 Angle of asymptote.

$n-m = 4$ ← no of asymptote

$$\phi = \frac{(2K+1)180}{n-m} \qquad K = 0, 1, 2, 3$$

$\phi = 45, 135, 225,$ and 315°

Step 4 Centroid.

$$\sigma = \frac{0-2-1-1}{4} = -1$$

step 5 the root locus branch on real axis is lies between 0 and -2 only.

Step 6

Breakaway points

$$\frac{dK}{ds} = 0$$

$$K = -s(s+2)(s^2+2s+4)$$
$$= s^4 + 4s^3 + 8s^2 + 8s$$

$$\frac{dK}{ds} = 4s^3 + 12s^3 + 16s + 8 = 0$$

the roots are $s = -1, -1 \pm j1$

$s = -1$ is the root locus lying on real axis.

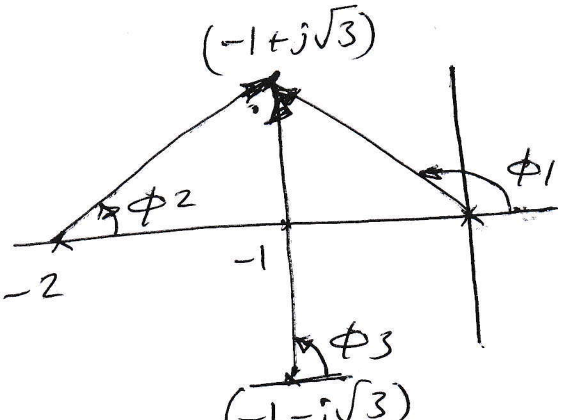
test for $-1 \pm j1$

$$\angle G(s)H(s)\Big|_{s=-1+j1} = \text{angle of } \frac{K}{(-1+j1)(-1+j+2)\{(-1+j)^2 + 2(-1+j) + 4\}}$$

$$= (-135 - 45 - 0)$$
$$= -18^\circ$$

This angle is satisfied, therefore $s = -1 \pm j1$ will be points on the root locus.

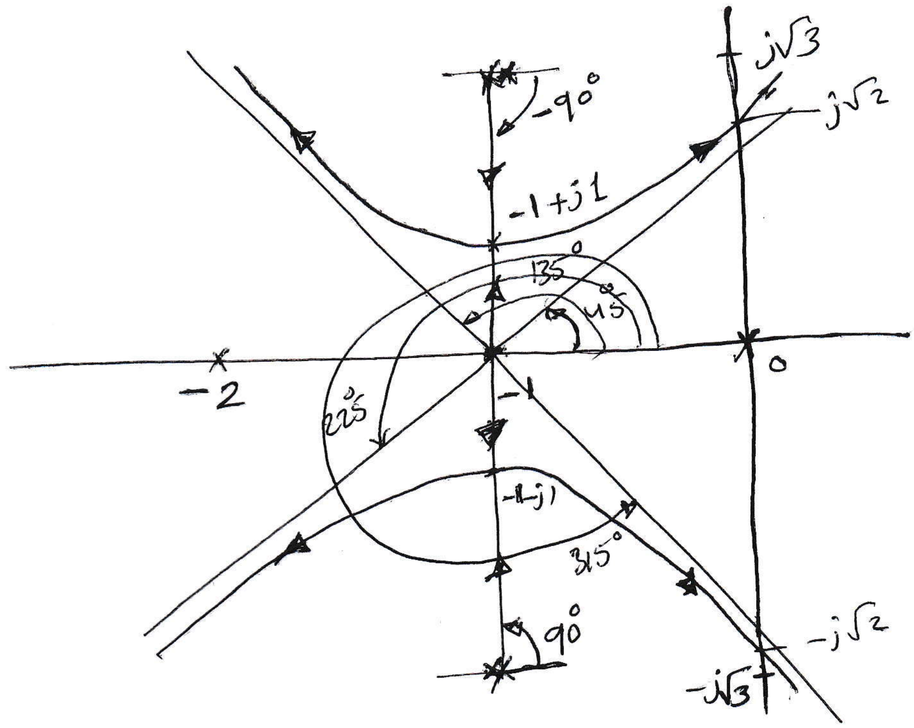
Step 7 Angles of departure



the angle at the complex pole

$s = -1 + j\sqrt{3}$

$$\phi = -\phi_1 - \phi_2 - \phi_3$$
$$= -120 - 90 - 60 = -270$$
$$\phi_P = 180 - 270^\circ = -90^\circ$$



Step 8

Crossing of $j\omega$-axis

the characteristic equation is

$$s^4+4s^3+8s^2+8s+K=0$$

By Routh's Criterion,

$$\begin{array}{c|ccc}
s^4 & 1 & 8 & K \\
s^3 & 4 & 8 & \\
s^2 & 6 & K & \\
s^1 & \frac{48-4K}{6} & & \\
s^0 & K & &
\end{array}$$

$$48-4K=0 \Rightarrow K=\frac{48}{4}=12$$

with $K=12$ to find the auxiliary equation is

$$6s^2+12=0 \Rightarrow s=\pm j\sqrt{2}$$

a) Let us find the value of K at $s = -1 + j$

$$K = -s(s+2)(s^2+2s+4)\big|_{s=-1+j}$$

$$K = -(-1+j)(-1+j+2)\left[(-1+j)^2+2(-1+j)+4\right]$$

$$K = 4$$

value of K at $s = -1$

$$K = -s(s+2)(s^2+\cancel{2s}+4)\big|_{s=-1}$$

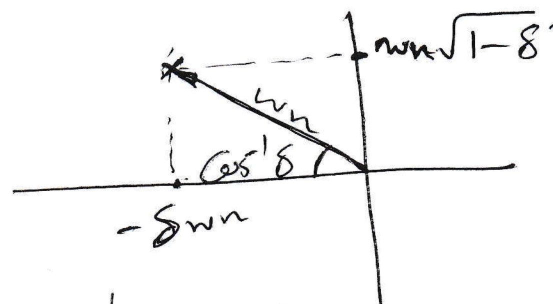
$$= 1(1)\cdot(1-2+4)$$

$$= 3$$

b)

the complex pole can be written as

$$\underbrace{-\delta\omega_n}_{\text{real}} + \underbrace{j\omega_n\sqrt{1-\delta^2}}_{\text{imaginary}}$$



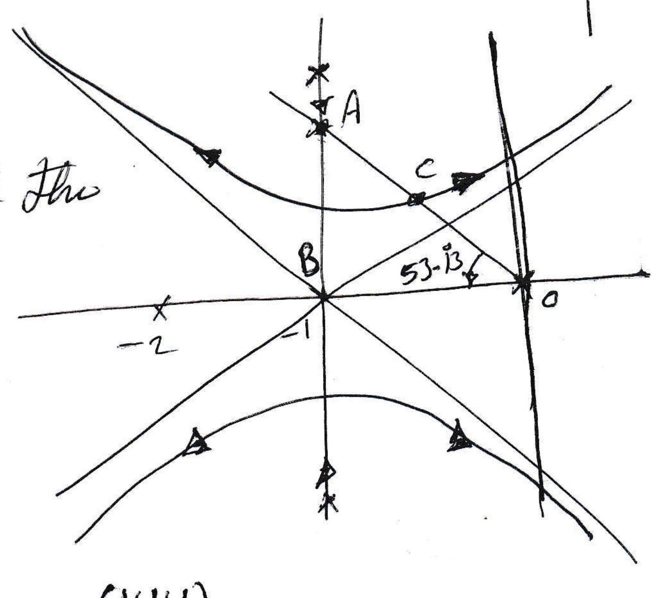
$\cos^{-1} 0.6 = 53.13^\circ$

$AB = OB \tan 53.13^\circ$

$= 1.333$

The two complex roots of the closed loop system are,

$s = -1 \pm j\,1.333$



the value of K at this point is

$$K = |(-1+j1.333)(-1+j1.333+2)(-1+j1.333+1+j\sqrt{3})(-1+j1.333+1-j\sqrt{3})|$$

$$= 1.666 * 1.666 * 3.06 * 0.399$$

$$= 3.39$$

the characteristic equation is

$$s^4 + 4s^3 + 8s^2 + 8s + 3.39 = 0$$

$$s = -1 \pm j1.333$$

$$(s+1)^2 + 1.777$$

$$s^2 + 2s + 2.7777$$

Dividing the characteristic equation by this factor, we get the other two poles, the the other factor due to the other two poles, the factor is

$$s^2 + 2s + 1.233$$

the roots of this factor are

$$s = -1 \pm j0.472$$

the closed loop poles are

$$s = -1 + j1.333, \; -1 \pm j0.472$$

Ex obtain the root locus of a unity feedback system with

$$G(s) = \frac{K(s+4)}{s^2+2s+2}$$

(1)

Zero : $s = -4$

poles : $s = -1 \pm j1$

(2) there are two root locus branches starting at $-1 \pm j1$ and one branch on the finite zero $s = -4$ and the other on zero at infinity.

(3)

$n - m = 1$ one asymptote

angle of asymptotes $\phi = 180^\circ$

(4) Centroid

$$\sigma = \frac{-1-1+4}{1} = 2$$

(5) Root locus branches on real axis

(6) Breakaway point

$$1 + \frac{K(s+4)}{s^2+2s+2} = 0$$

$$s^2+2s+2 + K(s+4) = 0$$

$$K = -\frac{s^2+2s+2}{s+4}$$

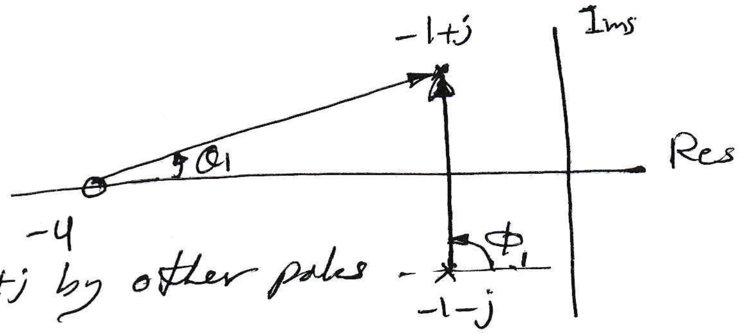
$$\frac{dK}{ds} = \frac{(s+4)(2s+2) - (s^2+2s+2)}{(s+4)^2} = 0$$

$$s^2+8s+6=0 \Rightarrow s_{1,2}=-0.837, -7.162$$

$s_1=-0.837$ is not a point on the root locus

$\therefore$ the breakin point is $s=-7.162$

(7) angle of departure

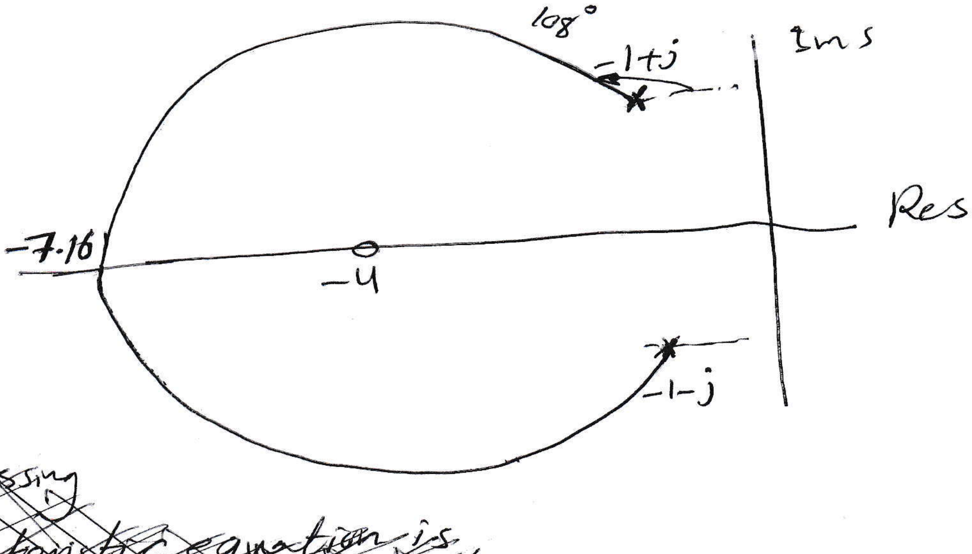


Angle contribution at $-1+j$ by other poles and zero

$$\phi=-90+\tan^{-1}\frac{1}{3}$$
$$=-71.56^\circ$$

the angle of departure from $(-1+j)$

$$\phi_p=180-71.56^\circ$$
$$=108.43^\circ$$



~~Step 7 jω-axis crossing~~

~~the characteristic equation is~~

~~$s^3+s^2(K+4)+2Ks+2K=0$~~

~~$2K^2+6K=0 \Rightarrow K=0, K=-3$ not valid~~

$\underline{\underline{EX}}$ sketch the root locus for

$$G(s)\,H(s) = \frac{K(s^2+2s+2)}{s^2(s+4)}$$

① Zeros: $-1 \pm j$

poles: $s = 0, 0, -4$

② there are 3 root locus branches, Two of them approach the Zeros at $-1+j1$. the third goes to infinity along the asymptote.

③ $n - m = 3 - 2 = 1$ one asymptote

angle of asymptote $\Rightarrow \theta = 180^\circ$

$$\sigma = \frac{-4+1+1}{1} = 2$$

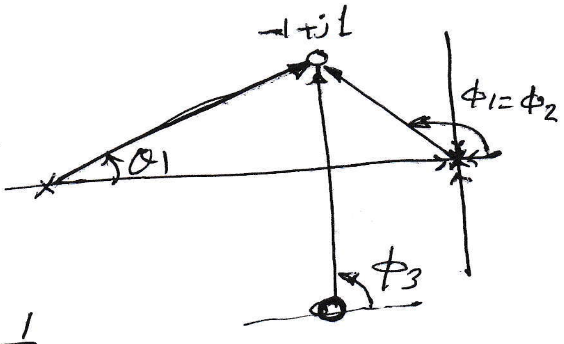
④ Root locus on real axis lies between $-\infty$ and -4

⑤ Breakaway point

we have multiple root at $s=0 \Rightarrow$ breakaway at $s=0$

⑥ angle of arrival at complex zero $-1 \pm j1$

Total angle contribution by all other poles and Zeros at $s = -1 \pm j$



$$\phi = -2(180 - \tan^{-1} 1) + 90 - \tan^{-1}\frac{1}{3}$$
$$= -198.43^\circ$$

Angle of arrival at Zero at $-1 \pm j1$ is

$$\phi_z = 180 - (-198.43)$$
$$= 18.43^\circ$$

⑦ $j\omega$-axis Crossing

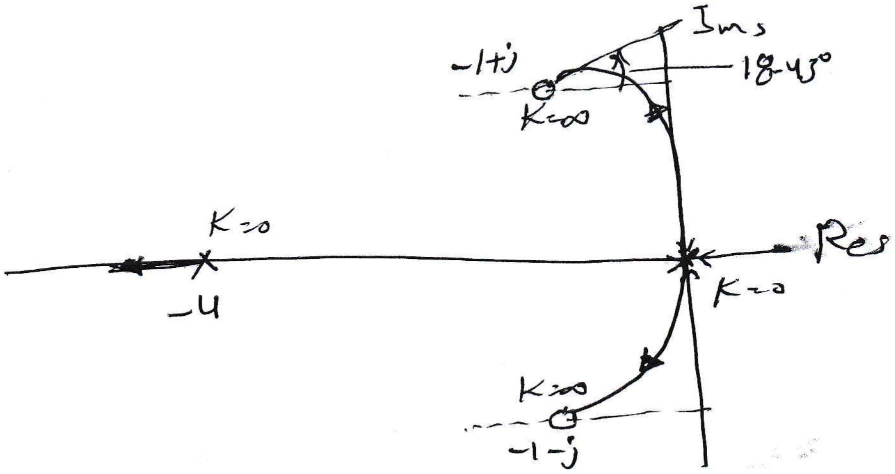
The characteristic equation is

$$s^3 + s^2(K+4) + 2Ks + 2K = 0$$

s^3	1	$2K$
s^2	$K+4$	$2K$
s^1	$\dfrac{2K^2+8K-2K}{K+4}$	0
s^0	$2K$	

$$2K^2 + 6K = 0 \Rightarrow K = 0,\ K = -3$$

$K = -3$ is not valid, $K = 0 \Rightarrow s = 0,\ j\omega = 0$



EX sketch the root locus of

$$G(s)H(s) = \frac{K(s+1)}{s^2(s+2)(s+4)}$$

① Zeros: -1

poles: $0, 0, -2, -4$

② there are 4 roots locus branches, one of them reaches $s = -1$, and the other three branches go to zero at infinity.

③ $n - m = 4 - 1 = 3$, three asymptotes.

angle of asymptote $\phi = 60°, 180°, -60°$

④ centroid $\sigma = \dfrac{-2-4+1}{3} = -\dfrac{5}{3}$

(5) Root locus on real axis between -1 and -2, and -4 to $-\infty$

(6) Breakaway point

$$1 + \frac{K(s+1)}{s^2(s+2)(s+4)} = 0$$

$$s^4 + 6s^3 + 8s^2 + Ks + K = 0$$

$$K = -\frac{s^2(s+2)(s+4)}{(s+1)} = -\frac{(s^4+6s^3+8s^2)}{s+1}$$

$$\cancel{\frac{dK}{ds} =}$$

$$\frac{dK}{ds} = -\left[\frac{(s+1)*(4s^3+18s^2+16s) - (s^4+6s^3+8s^2)*1}{(s+1)^2}\right]$$

$$= -\left\{\frac{4s^4+18s^3+16s^2+4s^3+18s^2+16s - s^4 - 6s^3 - 8s^2}{(s+1)^2}\right\}$$

$$= -\left[\frac{3s^4+12s^3+30s^2+16s}{(s+1)^2}\right] = 0$$

$$s(3s^3+12s^2+30s+16) = 0$$

$$s = 0$$

(7) There are no complex poles or zeros angle of arrival or departure.

8) $j\omega$-axis crossing

Routh table

s^4	1	8	K
s^3	6	K	
s^2	$\frac{48-K}{6}$	K	
s^1	$\frac{(48K-K^2)/6-6K}{(48-K)/6}$	0	
s^0	K		

$48K - K^2 - 36K = 0$

$12K - K^2 = 0 \Rightarrow K(12-K) \Rightarrow K=0$ or $K=12$

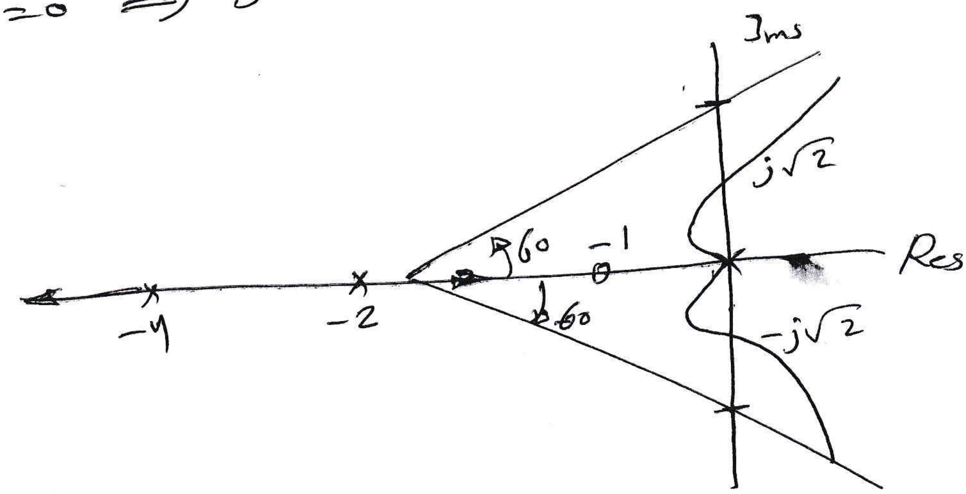
$K=0 \Rightarrow s=0$

$K=12$

by auxiliary equation

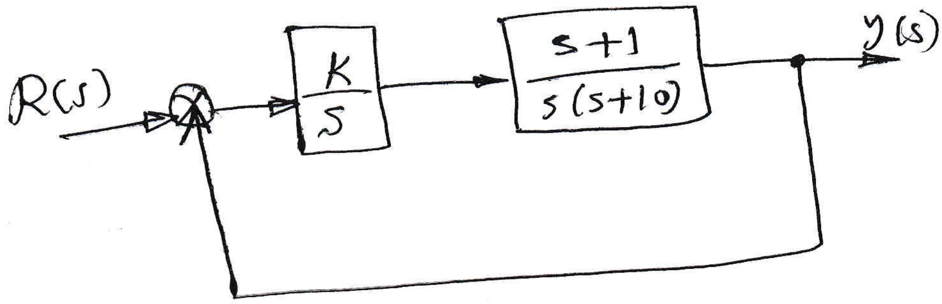
$\frac{48-12}{6}s^2 + 12 = 0$

$36s^2 + 72 = 0 \Rightarrow s^2 = -2 \Rightarrow s = \mp j\sqrt{2}$



EX

sketch the root Locus of the system shown below.



$$G(s)\,H(s) = \frac{K(s+1)}{s^2(s+10)}$$

① Zeros : -1

poles : $0, 0, -10$

② 3 root locus branches start from open loop poles and one ~~two~~ the other branch goes to the open loop Zero at $s=-1$ - the other two branches go to infinity.

③ $n-m = 3-1 = 2$ asymptotes

2 angles

$\phi = 90, -90$

④ Centroid $\quad \sigma = \dfrac{-10+1}{2} = -4.5$

⑤ the root locus on real axis lies between, -10 and -1.

⑥ breakaway points

$$K = -\frac{s^2(s+10)}{s+1}$$

$$\frac{dK}{ds} = -\frac{(s+1)(3s^2+20s) - s^2(s+10)}{(s+2)^2} = 0$$

$2s^3 + 13s^2 + 20s = 0$

$s = 0, -2.5$ and -4

all them are the breakaway points.

at $s=0 \Rightarrow K=0$

at $s=-2.5$

$$K=-\frac{s^2(s+10)}{s+1}\Bigg|_{s=-2.5}$$

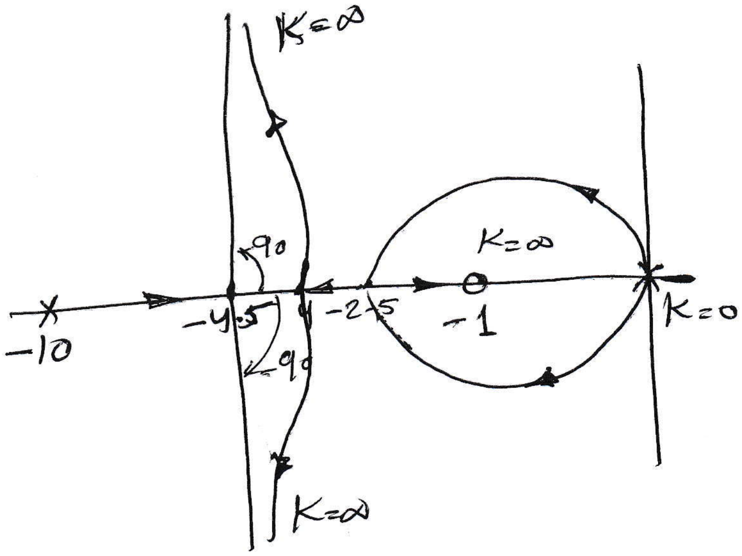
$$K=-\frac{6.25*7.5}{-1.5}$$

$$=31.25$$

at $s=4$

$$K=-\frac{16*6}{-3}$$

$$=32$$



Root Locus

Homework 5

1) Draw the root locus plot of the system with the following open loop transfer function, with unity feed back. Determine the

a) centroid,

b) angle of asymptotes,

c) breakaway if any

d) Angles of departure/arrival, if any

e) value of K, if any, for $j\omega$-axis crossing and frequency of sustained oscillations for this value of K.

i) $\dfrac{K}{s(s+4)(s+1)}$ ii) $\dfrac{K(s+1)}{s(s+4)(s+11)}$ iii) $\dfrac{K}{(s+2)(s^2+s+2)}$

iv) $\dfrac{K(s+1)}{s(s-1)}$

2) Determine the break away points, angles of departure and centroid of the root locus for the system

$$G(s)\,H(s) = \frac{K(s+3)}{s(s+5)(s+6)(s^2+2s+2)}$$

Also sketch the root locus

3) sketch the root locus of

$$G(s)\,H(s) = \frac{K(s+1)}{s^2(s+4)}$$

with unity feedback.

4) sketch the root locus of

$$G(s)H(s) = \frac{1}{s(s+2)[(s+1)^2+4]}$$

5) sketch the root locus of

a) $G(s)H(s) = \dfrac{K}{s^2(s+8)}$

b) $G(s)H(s) = \dfrac{K(s+3)}{s^2(s+8)}$

c) $G(s)H(s) = \dfrac{(s+1)^2}{s^3(s+4)}$